INTERACTIONS OF STRINGS AND EQUIVARIANT HOMOLOGY THEORIES
SHINGO OKUYAMA AND KAZUHISA SHIMAKAWA

Abstract. We introduce the notion of the space of parallel strings with partially summable labels, which can be viewed as a geometrically constructed group completion of the space of particles with labels. We utilize this to construct a machinery which produces equivariant generalized homology theories from such simple and abundant data as partial monoids.

1. Introduction

In [6] we attached to any pair of a Euclidean space $V$ and a partial abelian monoid $M$ a space $C(V, M)$ whose points are pairs $(c, a)$, where $c$ is a finite subset of $V$ and $a$ is a map $c \to M$, but $(c, a)$ is identified with $(c', a')$ if $c \subset c'$, $a'|c = a$, and $a'(v) = 0$ for $v \notin c$. Suppose that $V$ is an orthogonal $G$-module and $M$ admits a $G$-action compatible with partial sum operations for some finite group $G$. Then $C(V, M)$ is a $G$-space with respect to the $G$-action $g(c, a) = (gc, gag^{-1})$, $g \in G$.

Following Okuyama [3], let $I(\mathbb{R})$ denote the space of intervals in the real line, whose points are all the bounded submanifolds (i.e. finite unions of bounded intervals) of $\mathbb{R}$. Then $I(\mathbb{R})$ is a partial abelian monoid with respect to superimposition, and we can define

$$I(V, M) = C(V, I(\mathbb{R}) \land M)$$

for any partial abelian monoid with $G$-action $M$. The elements of $I(V, M)$ are represented by triples $(c, P, a)$, where $c$ is a finite subset of $V$, $P$ is a union of parallel intervals of the form $\bigcup_{v \in c} \{v\} \times P(v) \subset V \times \mathbb{R}$, $P(v) \in I(\mathbb{R})$, and $a$ is a map $c \to M$.

The aim of this paper is to show that if $V$ is sufficiently large then there is a $G$-equivariant group completion map $C(V, M) \to I(V, M)$, and that the correspondence $X \mapsto \{\pi_n I(V, X \land M); n \geq 0\}$ extends to an $RO(G)$-graded generalized homology theory.

To state the precise form of the main results, let $\text{Top}(G)$ denote the category of all pointed $G$-spaces and pointed maps, with $G$ acting on morphisms (i.e. pointed maps) by conjugation. As we showed in [5], to any $G$-equivariant continuous functor $T$: $\text{Top}(G) \to \text{Top}(G)$ such that $T(\ast) = \ast$ there are associated pairings

$$X \land TY \to T(X \land Y), \quad TX \land Y \to T(X \land Y)$$

natural in both $X$ and $Y$. In particular, $T$ preserves $G$-homotopies.

Date: September 24, 2004.

2000 Mathematics Subject Classification. 55N20, 55N91, 55P47.
Suppose $V$ is linearly and equivariantly isometric to the direct product of countably many copies of the regular representation of $G$ over the real number fields. Such a $G$-module $V$ is said to be sufficiently large. Now, the main results can be stated as follows.

**Theorem 1.** There is a diagram of Hopf $G$-spaces

$$C(V, M) \xrightarrow{\lambda} I_+(V, M) \xrightarrow{\rho} I(V, M)$$

in which $\lambda$ and $\rho$ are maps of Hopf $G$-spaces natural in $M$, and satisfy the following:

1. $\lambda$ is a $G$-homotopy equivalence.
2. $\rho$ is a $G$-equivariant group completion, that is to say, $\rho^H: I_+(V, M)^H \to I(V, M)^H$ is a group completion for every subgroup $H$ of $G$.

**Theorem 2.** The correspondence $X \to I(V, X \wedge M)$ is a $G$-equivariant continuous functor of $\text{Top}(G)$ into itself and we have

1. For any orthogonal $G$-module $W$ the natural map
   $$I(V, X \wedge M) \to \Omega^W I(V, \Sigma^W X \wedge M)$$
   adjoint to $S^V \wedge I(V, X \wedge M) \to I(V, S^W \wedge X \wedge M)$ is a weak $G$-equivalence.
2. There exists an $RO(G)$-graded homology theory $h^G_*(-)$ such that
   $$h^G_n(X) = \pi_n I(V, X \wedge M)^G$$
   holds for any $X$ and $n \geq 0$.

These theorems enable us to construct equivariant generalization of the popular homology theories. For example, consider the simplest case $M = S^0$. Then $C(V, X)$ is the usual configuration space, and hence its group completion $I(V, X)$ is weakly $G$-equivalent to the equivariant infinite loop space $\Omega^V \Sigma^V X$ by [1, Theorem (1.18)]. Thus we obtain the $G$-equivariant stable homotopy theory in this case. On the other hand, if we take arbitrary positive numbers as labels then we obtain an $RO(G)$-graded homology theory extending the ordinary homology $\tilde{H}_n(X/G, \mathbb{Z})$. (Compare [2].) $K$-theory type examples also occur from our method, which will be discussed in a future paper.

## 2. Partial abelian monoids with $G$-action

**Definition 3.** A pointed $G$-space $M$ is called a partial abelian monoid with $G$-action, or $G$-partial monoid for short, if there are $G$-invariant subspaces $M_n$ of $M^n$ ($n \geq 0$) and $G$-equivariant maps, called partial sum operations,

$$M_n \to M, \quad (a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n$$

satisfying the conditions below.

1. $M_0 = \{0\}$.
2. $M_1 \to M$ is the identity of $M$. 


3) Let \((a_1, \ldots, a_n) \in M^n\) and let \(J_1, \ldots, J_r\) be pairwise disjoint subsets of \(\{1, \ldots, n\}\) such that \(J_1 \cup \cdots \cup J_r = \{1, \ldots, n\}\) holds. Suppose that the partial sum \(\sum_{j \in J_i} a_j\) exists for every \(k, 1 \leq k \leq r\). Then \((a_1, \ldots, a_n) \in M^n\) holds if and only if \(\left(\sum_{j \in J_1} a_j, \ldots, \sum_{j \in J_r} a_j\right) \in M_r\), and in that case we have

\[
\sum_{1 \leq j \leq n} a_j = \sum_{j \in J_1} a_j + \cdots + \sum_{j \in J_r} a_j
\]

Among the examples we have

1) Let \(A\) be a topological abelian monoid with \(G\)-action. Then any \(G\)-invariant subset \(M\) of \(A\) with \(0 \in M\) can be regarded as a \(G\)-partial monoid by taking \(M^n = \{(a_1, \ldots, a_n) \in M^n \mid a_1 + \cdots + a_n \in M\}\).

2) Any pointed \(G\)-space \(X\) is a \(G\)-partial monoid with respect to the trivial partial sum operations, i.e., folding maps \(X_n = X \vee \cdots \vee X \to X\). In fact this is a special case of the previous example, as \(X\) is a \(G\)-invariant subset of the infinite symmetric product \(SP^\infty X\).

3) Let \(V\) be an infinite dimensional real inner product space on which \(G\) acts through linear isometries. Then the Grassmannian \(Gr(V)\) of finite-dimensional subspaces of \(V\) is a \(G\)-partial monoid with respect to the inner direct sum operations

\[
Gr(V)_n = \{(W_1, \ldots, W_n) \mid W_i \perp W_j, \ i \neq j\} \underbrace{\oplus}_{M} Gr(V)
\]

**Definition 4.** For given \(G\)-partial monoids \(M\) and \(N\), their smash product \(M \wedge N\) is a \(G\)-partial monoid whose partial sums are generated by the distributivity relations:

\[
c_1 \wedge d + \cdots + c_k \wedge d = (c_1 + \cdots + c_k) \wedge d, \quad (c_1, \ldots, c_k) \in M_k
c \wedge d_1 + \cdots + c \wedge d_l = c \wedge (d_1 + \cdots + d_l), \quad (d_1, \ldots, d_l) \in N_l
\]

**Example 5.** If \(X\) is a pointed \(G\)-space then for any \(G\)-partial monoid \(M\), \(X \wedge M\) is a \(G\)-partial monoid such that

\[
(X \wedge M)_n = X \wedge M_n
\]

holds for every \(n \geq 0\).

For any orthogonal \(G\)-module \(V\), the labeled configuration space \(C(V, M)\) is a \(G\)-partial monoid with respect to the partial sum operations

\[
C(V, M)_n \to C(V, M), \quad ((c_1, a_1), \ldots, (c_n, a_n)) \mapsto (\bigcup_i c_i, \bigcup_i a_i).
\]

Here \(C(V, M)_n\) consists of those \(n\)-tuples \(((c_i, a_i)) \in C(V, M)^n\) such that for every \(x \in \bigcup_i c_i\) the sum \(\sum_{i \in \Lambda(x)} a_i\) exists, where \(\Lambda(x) = \{i \mid x \in c_i\}\), and \(\bigcup_i a_i\) denotes the map \(x \mapsto \sum_{i \in \Lambda(x)} a_i\). Furthermore, if \(V\) is sufficiently large then \(C(V, M)\) has a \(G\)-equivariant multiplication defined as follows.

Choose two distinct points \(v_0\) and \(v_1\) from \(V^G\), and define a multiplication \(\mu: C(V, M) \times C(V, M) \to C(V, M)\) to be the composite

\[
C(V, M) \times C(V, M) \xrightarrow{\langle i, j \rangle} C(V \times V, M)_2 \xrightarrow{\cup} C(V \times V, M) \xrightarrow{\beta} C(V, M)
\]
where \( i_* \) and \( j_* \) are induced by the respective embeddings \( v \mapsto (v, v_0) \) and \( v \mapsto (v, v_1) \), and \( l_* \) is induced by some \( G \)-equivariant linear isometry \( l : V \times V \to V \).

Since the space of \( G \)-linear isometries \( V^n \to V \) is contractible for every \( n \geq 1 \), and since \( \dim V^G = \infty \), \( \mu \) does not depend on the choices of \( v_0, v_1 \) and \( l \) up to \( G \)-homotopy. It is easy to see that

**Proposition 6.** \( C(V, M) \) is a homotopy associative and homotopy commutative Hopf \( G \)-space with multiplication \( \mu \) and with unit \( \emptyset \in C(V, M)^G \).

3. The space of parallel strings with labels

As usual, the symbols \([a, b], [a, b), (a, b], (a, b)\) represent bounded (closed, or half-open, or open) intervals, and \( b - a \) is called the length of the interval. We shall allow intervals to have length 0. Thus, half-open intervals of length 0 are identified with the points of \( \mathbb{R} \).

Following [3], we denote by \( I(\mathbb{R}) \) the space of intervals of the real line \( \mathbb{R} \). An element of \( I(\mathbb{R}) \) can be written as a union, say \( P = J_1 \cup \cdots \cup J_r \), of finite number of pairwise disjoint bounded intervals. Here we may suppose \( J_i < J_{i+1} \) holds for \( 1 \leq i < r \), that is, \( x \in J_i \) and \( y \in J_{i+1} \) yields \( x < y \). But the union \( J_i \cup J_{i+1} \) can be replaced by a single interval \( J \) if \( J_i \cup J_{i+1} = J \) is a connected interval, and \( J_i \) can be removed if \( J_i \) is a half-open interval of length 0. This means that half-open intervals are collapsible to the empty set. (Strictly speaking, our definition of \( I(\mathbb{R}) \) is slightly different from the original one given in [3] in that the closed intervals of length 0, i.e. the points, are allowed.)

Let \( I(\mathbb{R})_+ \) be the subset of \( I(\mathbb{R}) \) consisting of such \( J_1 \cup \cdots \cup J_r \) that each \( J_i \) is a closed interval (or, a point). Similarly, let \( I(\mathbb{R})_- \) (resp. \( I(\mathbb{R})_0 \)) be the subset consisting of open (resp. half-open) intervals. Then \( I(\mathbb{R}) \) is a partial abelian monoid with respect to the superimposition,

\[
I(\mathbb{R})_n \to I(\mathbb{R}) \quad (P_1, \ldots, P_n) \mapsto P_1 \cup \cdots \cup P_n,
\]

where \( (P_1, \ldots, P_n) \in I(\mathbb{R})_n \) if and only if \( P_i \cap P_j = \emptyset \) for \( i \neq j \). Clearly, \( I(\mathbb{R})_+ \), \( I(\mathbb{R})_- \), and \( I(\mathbb{R})_0 \) are partial submonoids of \( I(\mathbb{R}) \).

**Definition 7.** Given an orthogonal \( G \)-module \( V \) and a \( G \)-partial monoid \( M \) let

\[
I(V, M) = C(V, I(\mathbb{R}) \land M), \quad I_+(V, M) = C(V, I(\mathbb{R})_+ \land M).
\]

\( I(V, M) \) is called the space of parallel strings in \( V \) with labels in \( M \).

To relate \( I(V, M) \) with \( C(V, M) \), we introduce a map \( b : I(\mathbb{R})_+ \to C(\mathbb{R}) \) which takes \( J_1 \cup \cdots \cup J_r \) to the finite set \( \{ bJ_1, \ldots, bJ_r \} \) consisting of the barycenters of \( J_i \)'s.

**Lemma 8.** The natural map \( I_+(V, M) \to C(V, C(\mathbb{R}) \land M) \) induced by \( b : I(\mathbb{R})_+ \to C(\mathbb{R}) \) is a \( G \)-homotopy equivalence.

**Proof.** Let \( i : C(\mathbb{R}) \to I(\mathbb{R})_+ \) be the inclusion which identifies a finite subset of \( \mathbb{R} \) with a set of closed intervals of length 0. Then both \( ib \) and \( bi \) are homotopic to the identities through maps of partial monoids. Hence the induced map \( I_+(V, M) \to C(V, C(\mathbb{R}) \land M) \) is a \( G \)-homotopy equivalence. \( \square \)
Lemma 9. If $V$ is sufficiently large then the inclusion
$$C(V, C(\mathbb{R}) \wedge M) \to C(V \times \mathbb{R}, M)$$
is a $G$-homotopy equivalence.

Proof. Let $i: \mathbb{R} \to V^G$ be an linear embedding, and define a homotopy $h: I \times \mathbb{R} \to V$ by $h(t, x) = (1-t)i(x)$. If we write $h_t(x) = h(t, x)$ then $h_0 = i$ and $h_1$ is the constant map with value $0$. Let $l$ be a $G$-linear isometry $V \times V \to V$. Then there is a homotopy
$$H: I_+ \wedge C(V \times \mathbb{R}, M) \to C(V \times \mathbb{R}, M)$$
such that $H_t = H(t, -)$ is induced by the composite
$$V \times \mathbb{R} \xrightarrow{1 \times \text{diag.}} V \times \mathbb{R} \times \mathbb{R} \xrightarrow{1 \times h_t \times 1} V \times V \times \mathbb{R} \xrightarrow{1 \times l} V \times \mathbb{R},$$

One easily observes that
1. $H$ restricts to a map $I_+ \wedge C(V, C(\mathbb{R})) \wedge M) \to C(V, C(\mathbb{R}) \wedge M),$
2. The image of $H_0$ is contained in $C(V, C(\mathbb{R})) \wedge M)$, and
3. $H_1$ is induced by the linear isometry $l' \times 1: V \times \mathbb{R} \to V \times \mathbb{R}$, where $l'$ is the composite $V = V \times \{0\} \subset V \times V \xrightarrow{l} V$.

As the space of linear isometries of $V$ is contractible, $l'$ is equivariantly isotopic to the identity through $G$-linear isometries. Thus we have $H_0 \simeq H_1 \simeq 1$, hence $H_0$ induces a homotopy inverse to the inclusion $C(V, C(\mathbb{R}) \wedge M) \to C(V \times \mathbb{R}, M)$. □

Corollary 10. If $V$ is sufficiently large then there is a map of Hopf $G$-spaces
$$\lambda: I_+(V, M) \to C(V, M)$$
which is natural in $M$ and is a $G$-homotopy equivalence.

Proof. Choose a $G$-linear isometry $l: V \times \mathbb{R} \cong V$, and define $\lambda$ as the composite
$$I_+(V, M) \xrightarrow{b} C(V, C(\mathbb{R}) \wedge M) \xrightarrow{\subseteq} C(V \times \mathbb{R}, M) \xrightarrow{l} C(V, M)$$
□

4. Proof of Theorem 1

We have already constructed an equivalence of Hopf $G$-spaces $\lambda: I_+(V, M) \to C(V, M)$ natural in $M$.

Let $\rho: I_+(V, M) \to I(V, M)$ be the map induced by the inclusion $I(\mathbb{R})_+ \subset I(\mathbb{R})$. We need to show that for every subgroup $H$ the following holds:

$$(4.1) \quad \rho^H: I_+(V, M)^H \to I(V, M)^H$$ is a group completion

Observe that $V$ is an $H$-universe for any subgroup $H$ of $G$. Hence (4.1) for general $H$ follows from the special case $H = G$. But the usual argument using the notion of orbit type family enables us to reduce the proof of this problem to the case where $G$ is trivial. (For example, see [1].) Thus we may assume $G = 1$ and $V = \mathbb{R}^\infty$.

Recall from [6] that there is a weak equivalence of Hopf spaces
$$\Phi: D(M) \to C(\mathbb{R}^\infty, M)$$
from a $CW$-monoid $D(M)$ defined as follows.
Let \( Q(M) \) be the topological category whose space of objects is \( \coprod_{p \geq 0} M^p \), and in which a morphism from \((a_i) \in M^p \) to \((b_j) \in M^q \) is given by a map of finite sets \( \theta: \{1, \ldots, p\} \to \{1, \ldots, q\} \) such that \( b_j = \sum_{i \in \theta^{-1}(j)} a_i \) hold for \( 1 \leq j \leq q \). If we write \((b_j) = \theta \ast (a_j)\) then the morphism above can be written in the form \( S \to \theta \ast S \), where \( S = (a_j) \in M^p \). Now \( D(M) \) is defined to be the realization of the diagonal simplicial set \( [k] \mapsto S_k N_k Q(M) = N_k Q(S_k M) \) associated with the bisimplicial set of the total singular complex of the nerve of \( Q(M) \). As \( Q(M) \) is a permutative category with respect to the operation \((a_1, \ldots, a_p) \cdot (b_1, \ldots, b_q) = (a_1, \ldots, a_p, b_1, \ldots, b_q)\) \( D(M) \) is a homotopy commutative monoid with respect to the induced multiplication.

Since \( \Phi \) is natural in \( M \), Theorem 1 follows from

**Proposition 11.** The natural map \( D(I(\mathbb{R})_+ \wedge M) \to D(I(\mathbb{R}) \wedge M) \) induced by the inclusion \( I(\mathbb{R})_+ \subset I(\mathbb{R}) \) is a group completion.

The rest of this section is devoted to the proof of this proposition.

Given a map of topological monoids \( f: D \to D' \) let \( B(D, D') \) denote the realization of the category \( B(D, D') \) whose space of objects is \( D' \) and whose space of morphisms is the product \( D \times D' \), where \((d, d') \in D \times D' \) is regarded as a morphism from \( d' \) to \( f(d) \cdot d' \). Then there is a sequence of maps \( D' = B(0, D') \to B(D, D') \to B(D, 0) = BD \) induced by the maps \( 0 \to D \) and \( D' \to 0 \) respectively. Observe that \( BD \) is the standard classifying space of the monoid \( D \) and \( B(D, D) \) is contractible when \( f \) is the identity.

Let us take \( D = D(I(\mathbb{R})_+ \wedge M) \) and \( D' = D(I(\mathbb{R}) \wedge M) \). Let \( i: D \to D' \) be the map induced by the inclusion \( I(\mathbb{R})_+ \to I(\mathbb{R}) \). Then we have a commutative diagram

\[
\begin{array}{ccc}
D & \longrightarrow & B(D, D) \\
\downarrow & & \downarrow \text{B(1,i)} \\
D' & \longrightarrow & B(D, D')
\end{array}
\]

in which the upper and the lower sequences are respectively associated with the identity and the inclusion \( i: D \to D' \).

**Lemma 12.** The natural map \( D \to \Omega BD \) is a group completion.

This follows from the fact that \( D \) is a homotopy commutative, hence admissible, monoid.

**Lemma 13.** The lower sequence in the diagram (4.2) is a homotopy fibration sequence with contractible total space.

Proposition 11 can be deduced from this, because \( D \to D' \) is equivalent to the group completion map \( D \to \Omega BD \) under the equivalence \( D' \simeq \Omega BD \).
Proof of Lemma 13. As $D$ acts on $D'$ through homotopy equivalences, the diagram

$$\begin{array}{ccc}
D' & \longrightarrow & B(D, D') \\
\downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & B(D, 0)
\end{array}$$

is homotopy-cartesian by Proposition 1.6 of [4]. This implies that the lower sequence in the diagram (4.2) is a homotopy fibration sequence.

Let us denote $D_k = N_k Q(S_k(I(R) \land M))$ and $D'_k = N_k Q(S_k(I(R) \land M))$ for $k \geq 0$. Then $B(D, D')$ is obtained as the realization of the diagonal simplicial set

$$[k] \mapsto B(D, D')_k = N_k B(D_k, D'_k).$$

Hence $B(D, D')$ is a CW-complex whose 0-cells correspond to elements of $D'_0$. 1-cells to pairs from $D_1$ to $D'_1$, and so on. In particular, a pair consisting of $(S \to \theta, S) \in D_1$ and $(T \to \psi, T) \in D'_1$ determines a path in $B(D, D')$ joining $d_0 T$ to $d_1(\theta, S \cdot \psi, T)$.

Let $\|B(D, D')\|$ denote the thick realization of $[k] \mapsto B(D, D')_k$. We shall show that the natural map

$$q: \|B(D, D')\| \to \|B(D, D')\| = B(D, D')$$

is homotopic to the constant map. This implies that $B(D, D')$ is contractible, since $q$ is a homotopy equivalence. (See [4, Appendix A].)

Let $j: \|D'\| \to \|B(D, D')\|$ be the inclusion, where $\|D'\|$ denotes the thick realization of $[k] \mapsto D'_k$. Let $r: \|B(D, D')\| \to \|D'\|$ be the map induced by the functor $B(D, D') \to D'$, which takes a morphism $(d, d')$ to the identity of $d \cdot d'$. Clearly the composite $jr: \|B(D, D')\| \to \|B(D, D')\|$ is homotopic to the identity. Hence $q$ is homotopic to the constant map if so is $qj: \|D'\| \to B(D, D')$.

Let $\|D'\|_n$ denote the image of $\bigsqcup_{k \leq n} D'_k \times \Delta^k$ in $\|D'\|$, and let $q j_n: \|D'\|_n \to B(D, D')$ be the restriction of $q j$ to $\|D'\|_n$. We construct a null homotopy $h_n$ of $q j_n$, by successively extending $h_{n-1}$ for all $n \geq 0$.

Given a bounded open interval $J = (a, a + \epsilon)$, put

$$\tilde{J} = [a + \epsilon/4, a + \epsilon/2], \quad \hat{J} = (a, a + \epsilon/4) \cup [a + 3\epsilon/4, a + \epsilon)$$

so that we have $\tilde{J} \cup \hat{J} = (a, a + \epsilon/2] \cup [a + 3\epsilon/4, a + \epsilon)$. More generally, for given $P = J_1 \cup \cdots \cup J_r \in I(R)_-$, we put

$$\hat{P} = \hat{J}_1 \cup \cdots \cup \hat{J}_r \in I(R)_+, \quad \tilde{P} = \tilde{J}_1 \cup \cdots \cup \tilde{J}_r \in I(R)_-.$$  

Notice that there are continuous deformations $P \to \hat{P}$ induced by the evident deformations $J_i \to \hat{J}_i$, and also $\tilde{P} \cup \hat{P} \to \emptyset$ which collapses half-open intervals to the empty set.

Let $S = (P_j \land a_j) \in S_0(I(R) \land M)^p$ be an element of $\|D'\|_0 = D'_0$. Choose decompositions $P_j = P_{j+} \cup P_{j0} \cup P_{j-}$, where $P_{j+} \in S_0 I(R)_+$, $P_{j0} \in S_0 I(R)_0$, $P_{j-} \in S_0 I(R)_-$, and put

$$S_+ = (P_{j+} \land a_j), \quad S_- = (P_{j-} \land a_j), \quad S_{-0} = (P_{j-} \cup P_{j0} \land a_j).$$
Let \( [S] \in B(D, D') \) be the image of \( S \) under \( q_j0 \). Then there is a chain of paths
\[
[S] \xrightarrow{\mu} [S_{-0}] \xrightarrow{\gamma} [0^p] \xleftarrow{\nu} \emptyset,
\]
where \( "T_0 \xrightarrow{\alpha} T_1" \) means a path (not a map) such that \( \alpha(0) = T_0 \) and \( \alpha(1) = T_1 \), defined as follows:

1. \( \mu \) is the composite of paths
\[
[S_{-0}] \xrightarrow{\alpha} [S_{+} \cdot S_{-0}] \xrightarrow{\varphi} [S],
\]
where \( \alpha \) corresponds to the pair \( (S_{+}, S_{-0}) \) regarded as an element of \( D_{1} \times D'_{1} = B(D, D')_{1} \) via the degeneracy \( s_0 \), and \( \varphi \) corresponds to the arrow \( (S_{+} \cdot S_{-0} \rightarrow \nabla_{s}(S_{+} \cdot S_{-0}) = S) \in D'_{1} \) given by the map \( \nabla : \{ 1, \ldots, 2p \} \rightarrow \{ 1, \ldots, p \} \) such that \( \nabla(j) = \nabla(p + j) = j \) holds for \( 1 \leq j \leq p \).

2. \( \gamma \) is the composite
\[
[S_{-0}] \xrightarrow{\gamma_1} [\tilde{S}_{-0}] \xrightarrow{\gamma_2} [\nabla_{s}(\tilde{S}_{-} \cdot \tilde{S}_{-0})] \xrightarrow{\gamma_3} [0^p]
\]
where \( \tilde{S}_{-0} = (\tilde{P}_{j} \cup P_{j0}, a_j) \), \( \gamma_1 \) is induced by the deformations \( P_{j} \rightarrow \tilde{P}_{j} \) regarded as an element of \( S_{1}I(\mathbb{R}) \), \( \gamma_2 \) is induced by the sequence \( \tilde{S}_{-0} \rightarrow \tilde{S}_{-} \cdot \tilde{S}_{-0} \rightarrow \nabla_{s}(\tilde{S}_{-} \cdot \tilde{S}_{-0}) \), and \( \gamma_3 \) is induced by the deformations \( \tilde{P}_{j} \cup \tilde{P}_{j} \cup P_{j0} \rightarrow 0 \).

3. \( \nu \) is induced by the unique map \( \emptyset \rightarrow 1 \).

Thus we obtain a path \( \alpha(S) \) in \( B(D, D') \) joining \( S \) to the basepoint \( \emptyset \), and hence a null homotopy \( h_0 : \| D' \|_0 \times I \rightarrow B(D, D') \) of \( q_j0 \) given by \( h_0(S, t) = \alpha(S)(t) \).

We shall extend \( h_0 \) to a null homotopy over \( \| D' \|_1 \). Given an element \( \theta : S \rightarrow T \) of \( D'_{1} \) with \( S \in S_{1}(I(\mathbb{R}) \wedge M)^p \) and \( T = \theta, S \in S_{1}(I(\mathbb{R}) \wedge M)^p \), we denote by \( [\theta] \) the composite of the 1-cell \( I \rightarrow \| D' \|_1 \) corresponding to \( \theta \) with \( q_j1 : \| D' \|_1 \rightarrow B(D, D') \). Thus \( [\theta] \) is a path in \( B(D, D') \) joining \( [d_0S] \) to \( [d_1T] \). We also write \( T_{+} = \text{Int} T_{+} \in Q(S_{1}(I(\mathbb{R}) \wedge M)) \), that is, \( T_{+} = (\text{Int} R_{j}, b_{j}) \) if \( T_{+} = (R_{j}, b_{j}) \). Then we have a path diagram
\[
\begin{array}{ccc}
d_1[T] & \xrightarrow{\mu} & d_1[T_{-0}] \\
\uparrow{\theta} & & \uparrow{\psi} \\
d_0[S] & \xrightarrow{1_{-\nu}} & d_0[S \cdot 0^q]
\end{array}
\]
\[
\begin{array}{ccc}
1_{-\nu} \gamma_{-1} & & 1_{-1} \\
\uparrow{1_{-\nu}} & & \uparrow{1_{-\nu}} \\
d_0[S] & \xrightarrow{1_{-\nu}} & d_0[S \cdot 0^q]
\end{array}
\]
where \( \xi \) and \( \xi^0 \) are induced by the respective arrows
\[
S \cdot 0^q \rightarrow \nabla(\theta \cdot 1)_{s}(S \cdot 0^q) = T, \quad 0^p \cdot 0^q \rightarrow \nabla(\theta \cdot 1)_{s}(0^p \cdot 0^q) = 0^q
\]
and \( \psi \) is given by the chain
\[
[d_0 S \cdot d_1 T^+ \cdot (\mu_1 \cdot d_1 T^+) \rightarrow [d_0 S \cdot d_1 T^+] \rightarrow [d_1 T^+] = [(d_1 T_{0+} \cup d_1 T^+) \cdot d_1 T^+] \rightarrow [d_1 T_{0+} \cup d_1 T^+] \rightarrow [d_1 T_{0+} \cup d_1 T^+ \cdot d_1 T^+] \rightarrow [d_1 T_{0+} \cup d_1 T^+ \cdot d_1 T^+] \rightarrow [d_1 T_{0+} \cup d_1 T^+ \cdot d_1 T^+] .
\]

One easily observes that the diagram above determines a null homotopy of \( \partial D^0 \) extending the one already defined on \( \partial D^0 \) to \( \partial D^1 \).

By induction, we can extend the construction above to arbitrary \( n \). Suppose that for every \( \mathcal{D} \in D_k^0 \) with \( k < n \), there exists a null homotopy of the corresponding \( k \)-cell \( [\mathcal{D}] : \Delta^k \rightarrow B(D, D') \) given by a chain of the form
\[
[\mathcal{D}] \rightarrow [\mathcal{D}_0] \rightarrow [\mathcal{D}_{-0}] \rightarrow [0] \rightarrow \emptyset
\]
which is compatible with face operators.

Let \( \mathcal{D}' = (S(0) \stackrel{\theta_1}{\longrightarrow} S(1) \stackrel{\theta_2}{\longrightarrow} \cdots \stackrel{\theta_n}{\longrightarrow} S(n)) \) be an element of \( D'_n \). Then the diagram similar to (4.4), but \( d_0 S \) and \( d_1 T \) are replaced by
\[
d_0 \mathcal{D}' = (d_0 S(1) \stackrel{\theta_2}{\longrightarrow} \cdots \stackrel{\theta_n}{\longrightarrow} d_0 S(n)) \in D'_{n-1}
\]
and \( d_i S(0) \in D'_0 \) respectively, yields a null homotopy of the \( n \)-cell \( [\mathcal{D}] \) which extends the ones already defined on its faces \( [d_i \mathcal{D}] \). This implies that the null homotopy can be extended over \( \|D'\|_n \), hence completes the proof of Proposition 11.

5. PROOF OF THEOREM 2

By a simplicial pointed \( G \)-space we shall mean a simplicial object in the category of pointed \( G \)-spaces and basepoint preserving \( G \)-maps. If \( X_\bullet \) is a simplicial pointed \( G \)-space then the basepoints of \( X_n \), \( n \geq 0 \), form a subsimplicial object and the natural map \( \|X_\bullet\| \rightarrow \|X_\bullet\|/\|\ast\| = \|X_\bullet\|' \) is a \( G \)-homotopy equivalence. Clearly the natural maps \( \Delta^k \times T(X_n) \rightarrow \|X_\bullet\|' \) induce \( \Delta^k \times T(X_n) \rightarrow \|X_\bullet\|' \) for \( n \geq 0 \).

Let \( T : \text{Top}(G) \rightarrow \text{Top}(G) \) be a \( G \)-equivariant continuous functor. Then to any simplicial pointed \( G \)-space \( X_\bullet \) there is associated a \( G \)-map \( \|T(X_\bullet)\|' \rightarrow T(\|X_\bullet\|') \) induced by the maps
\[
\Delta^k \times T(X_n) \rightarrow \Delta^k \times T(X_n) \rightarrow T(\Delta^k \times X_n) \rightarrow T(\|X_\bullet\|') .
\]

Also, for any pointed finite \( G \)-set \( S \) and any pointed space \( X \) there is a \( G \)-map
\[
T(S \times X) \rightarrow \text{Map}_0(S, T(X))
\]
whose adjoint \( S \times T(S \times X) \rightarrow T(X) \) is induced by the pairing \( S \times T(S \times X) \rightarrow X \) which takes \( (s, s, x) \) to \( x \) and \( (s, t, x) \) \( (s \neq t) \) to the basepoint of \( X \).

The following plays a key roll in the proof of Theorem 2.

**Proposition 14.** Let \( T : \text{Top}(G) \rightarrow \text{Top}(G) \) be a \( G \)-equivariant continuous functor. Suppose \( T \) satisfies the following conditions.

- (C1) \( T(\ast) = \ast \).
- (C2) For any simplicial pointed \( G \)-space \( X_\bullet \), the natural map \( \|T(X_\bullet)\|' \rightarrow T(\|X_\bullet\|') \) is a \( G \)-homotopy equivalence.


(C3) For any $X$ and $Y$ the map $T(X \vee Y) \rightarrow T(X) \times T(Y)$ induced by the projections $X \vee Y \rightarrow X$ and $X \vee Y \rightarrow Y$ is a $G$-homotopy equivalence.

(C4) For any subgroup $H$ the natural map $T(G/H \wedge X) \rightarrow \text{Map}(G/H, T(X))$ is a $G$-homotopy equivalence.

Suppose further that $T(X)^H$ is grouplike for any $X$ and any subgroup $H$ of $G$. Then we have

(a) For any orthogonal $G$-module $W$ the natural map $T(X) \rightarrow \Omega^W T(\Sigma^W X)$ adjoint to $S^W \wedge T(X) \rightarrow T(S^W \wedge X)$ is a weak $G$-homotopy equivalence.

(b) The correspondence $X \mapsto \{\pi_n T(X)^G\}$ is extendible to an RO($G$)-graded equivariant homology theory defined on the category of pointed $G$-spaces.

Proof. For any pointed $G$-space $X$ let $E(X) = \Omega T(\Sigma X)$. If $T$ satisfies (C1), (C2) and (C3) then by the equivariant version of [6, Theorem 2.12] we see that the natural map $T(X) \rightarrow E(X)$ is a $G$-equivariant group completion, and the sequence

$$E(A) \rightarrow E(X) \rightarrow E(X \cup CA)$$

associated with a pair of pointed $G$-spaces $(X, A)$ is a $G$-fibration sequence up to weak $G$-equivalence.

But $T(X) \rightarrow E(X) = \Omega T(\Sigma X)$ is a weak $G$-equivalence in our case, because $T(X)^H$ is grouplike for any subgroup $H$. Hence

$$T(A) \rightarrow T(X) \rightarrow T(X \cup CA)$$

is a $G$-fibration sequence up to weak $G$-equivalence. Moreover, it follows by the property of $G$-equivariant continuous functors that $T$ preserves $G$-homotopies. Thus the correspondence $X \mapsto \{\pi_n T(X)^G\}$ determines a $Z$-graded equivariant homology theory.

Let $\Gamma_G$ be the full subcategory of $\text{Top}(G)$ consisting of all finite pointed $G$-sets. To prove (a) and (b) we need only show that the correspondence $S \mapsto T(S \wedge X)$ from $\Gamma_G$ to $\text{Top}(G)$ is a special $\Gamma_G$-space in the sense of [5]. But this follows from the conditions (C3) and (C4).

Now let $T(X) = I(V, X \wedge M)$. We shall show that $T$ satisfies all the conditions (C1), (C2), (C3) and (C4). This of course implies Theorem 2.

Obviously (C1) holds, and (C2) is proved by the argument similar to that used in the proof of [6, Theorem 3.2].

To see that (C3) holds, let $T(X) \times T(Y) \rightarrow T(X \vee Y)$ be the composite

$$I(V, X \wedge M) \times I(V, Y \wedge M) \xrightarrow{(i_*, j_*)} I(V, (X \vee Y) \wedge M)^2 \xrightarrow{\mu} I(V, (X \vee Y) \wedge M)$$

where $i_*$ and $j_*$ are induced by the respective inclusions of $X$ and $Y$ into $X \vee Y$, and $\mu$ is the multiplication of Hopf $G$-space $I(V, (X \vee Y) \wedge M)$. By using the fact that the space of $G$-linear isometries of $V$ is contractible, one can show that the map above gives a $G$-homotopy inverse to $T(X \vee Y) \rightarrow T(X) \times T(Y)$.

Finally, to verify (C4) we shall construct a $G$-homotopy inverse to the natural map $T(G/H \wedge X) \rightarrow \text{Map}(G/H, T(X))$ by the following procedure:

(1) Choose a $G$-embedding $G/H \rightarrow V$ and a $G$-linear isometry $l : V \times V \rightarrow V$. 

(2) For given \( f : G/H \to T(X) \) let us write
\[
f(gH) = (c(gH), P(gH) \wedge a(gH))
\]
where \( c(gH) \subseteq V \), \( P(gH) : c(gH) \to I(\mathbb{R}) \) and \( a(gH) : c(gH) \to X \wedge M \).

(3) Let \( \tilde{c} \) be the image of the union \( \bigcup \{gH\} \times c(gH) \) under the embedding
\[
\iota : G/H \times V \subset V \times V \overset{1}{\to} V
\]

(4) Define \( \tilde{a} : \tilde{c} \to I(\mathbb{R}) \wedge G/H_+ \wedge X \wedge M \) by
\[
\tilde{a}(\iota(gH, \xi)) = P(gH) \wedge gH \wedge a(gH)(\xi), \quad \xi \in c(gH)
\]

(5) Define \( \rho : \text{Map}(G/H, T(X)) \to T(G/H_+ \wedge X) \) by \( \rho(f) = (\tilde{c}, \tilde{a}) \).
Again, that \( \rho \) gives a \( G \)-homotopy inverse to \( T(G/H_+ \wedge X) \to \text{Map}(G/H, T(X)) \) follows from the contractibility of the space of \( G \)-linear isometries of \( V \).

REFERENCES


Takuma National College of Technology
E-mail address: okuyama@dc.takuma-ct.ac.jp

DEPARTMENT OF MATHEMATICS, OKAYAMA UNIVERSITY, OKAYAMA 700, JAPAN
E-mail address: kazu@math.okayama-u.ac.jp